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Letter to the Editor

Stochastic stability of torsion oscillations in moving thin elastic bands

P. Kozić*, R. Pavlović

Department of Mechanical Engineering, University of Niš, P.O. Box 209, 18000 Niš, Serbia and Montenegro

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1. Introduction

Elastic band oscillations belong to the class of problems that can be generally considered as oscillatory problems of axially moving elastic bodies. This class of problems includes: oscillations of moving filaments in textile industry machines, oscillations of band saws, oscillatory movement of magnetic bands in audio devices, etc. Such systems are modelled by axially moving strings, beams, or plates. A great number of researchers have studied the problems of the moving thin elastic band. Nelson [1] analyzed deterministic transverse oscillations of a thin moving band and derived relations for determining natural frequencies of this system. Alspaugh [2] explored torsion oscillations of a thin rectangular moving band which moves at constant speed in the longitudinal direction. Soler [3] studied the stability of connected transverse and torsional oscillations of a thin moving band. Ariaratnam and Asokanthan [4] explored torsion oscillations of a moving band stretched at the ends by a harmonically variable force. They determined the analytical conditions of stability, and presented them graphically. Wang [5] analyzed a non-linear model of axially moving bands with end curvatures. Both analytical and numerical results show the impact of velocity, tensile force and bending radius at the band's ends upon the system's dynamic behaviour. Sund and Fung [6] propose a new dynamic model describing the behaviour between the fluid film and elastic deformation in a thin foil bearing. The string and beam models describing the magnetic tape are compared. Tylikowski [7] determined the conditions for uniform stochastic stability of a moving band by using Lyapunov's direct method, while Kozin and Milsted [8] determined the conditions for almost-sure asymptotic stability of a moving band by using the method developed by the authors. Kozić and Pavlović [9] determined the conditions of stochastic stability of a thin moving elastic strip when it is subjected to the parameters of random excitation that are wide-band stochastic processes of small intensity. By combining the Khas'minskii method of stochastic averaging and the procedure for determining the largest Lyapunov exponent, expressions for the largest Lyapunov exponent with respect to the system parameters have been determined.

*Corresponding author.
E-mail addresses: kozicp@yahoo.com (P. Kozić), ratpav@yahoo.com (R. Pavlović).

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In this paper, the differential equation of torsion oscillations of a moving elastic band is discretized by using Galerkin' method. In the opinion of the authors though, the tensile force is randomly varied. The differential equation of motion is substituted by an equivalent system of two Itô's differential equations of the first order. The differential equations are formed by the first and second statistic moments. By using Hurwitz's criterion, the conditions of stochastic stability of the thin moving elastic band are determined. The results obtained have been checked by applying Lyapunov's exponent.

2. Discretisation of the equation of motion

The torsional motion of a moving band shown in Fig. 1 will now be considered. Let the band be supported by frictionless rollers at span L, possess mass ρ per unit length. Assuming that each filamentary element of the band moves longitudinally at uniform speed c, the angular velocity of any point on the band is given by $(c\theta_x + \theta_t)$, where θ is the angle of twist. In the absence of any external load on the band, the equation representing the torsional motion of the moving band with the rectangular cross-section takes the following form (see Ref. [2]).

$$L(\theta) = \frac{\partial^2\theta}{\partial t^2} + \beta' \frac{\partial\theta}{\partial t} + (c^2 - c_0^2)\frac{\partial^2\theta}{\partial x^2} + 2c\frac{\partial^2\theta}{\partial x \partial t} = 0. \tag{1}$$

In Eq. (1), c_0^2 contains the effects of band tension on the torsional rigidity; it also consists of variable random tension in the band,

$$c_0^2 = 4\frac{G}{\rho}\frac{b^2}{h^2} + \frac{\sigma_0}{\rho}[1 + f(t)],$$

where G is the shear modulus of plate material, b the band thickness, h the band width, σ_0 the stress due to initial band tension, β' the damping coefficient. For the sake of a more concise representation, the equations will be non-dimensionalized by introducing the following quantities: $\xi = x/L,\ \tau = ct/L,\ \phi = h\theta/b,\ 2\beta = \beta' L/c.$

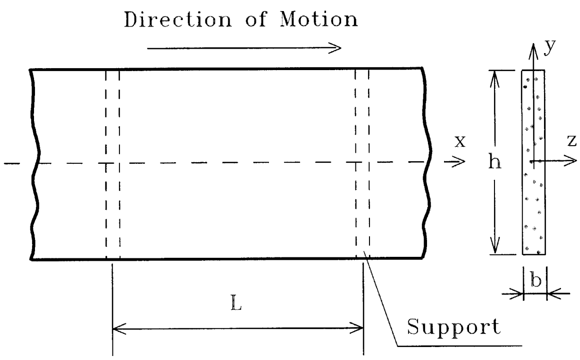

Fig. 1. Scheme of an axially moving thin elastic band.

With these definitions, Eq. (1) becomes

$$L(\phi) = \frac{\partial^2\phi}{\partial\tau^2} + 2\beta\frac{\partial\phi}{\partial\tau} + \left[4\frac{G}{\rho c^2}\frac{b^2}{h^2} + \frac{\sigma_0}{\rho c^2} - 1 + \frac{\sigma_0}{\rho c^2}f(\tau)\right]\frac{\partial^2\phi}{\partial\xi^2} + 2\frac{\partial^2\phi}{\partial\xi\partial\tau} = 0. \quad (2)$$

In order to simplify Eq. (2) further, use will be made of Galerkin's method to reduce this equation to a corresponding ordinary differential equation representing only the time-varying part of the solution. For a trial function, an infinite Fourier sine series will be chosen

$$\phi(\xi,\tau) = \sum_{n=1}^{\infty}\Phi_n(\tau)\sin n\pi\xi. \quad (3)$$

It can immediately be noticed that the trial function (3) is, in fact, admissible since it satisfies the boundary conditions at $\xi = 0, 1$ given by

$$\phi = 0, \quad \frac{\partial^2\phi}{\partial\xi^2} = 0.$$

Furthermore, by Galerkin's method it is required that

$$\int_0^1 L(\phi)\delta\phi \,\mathrm{d}\xi = 0. \quad (4)$$

By substituting Eq. (3) into Eq. (4) and evaluating the integral as indicated, it follows that the given trial function (3) will satisfy for each n

$$\ddot{\Phi}_n(\tau) + 2\beta\dot{\Phi}_n(\tau) + \{[\omega^{(n)}]^2 + g^{(n)}(\tau)\}\Phi_n(\tau) = 0, \quad (5)$$

where

$$[\omega^{(n)}]^2 = (n\pi)^2\left[4\frac{G}{\rho c^2}\frac{b^2}{h^2} + \frac{\sigma_0}{\rho c^2} - 1\right], \quad g^{(n)}(\tau) = \frac{\sigma_0}{\rho c^2}(n\pi)^2 f(\tau). \quad (6)$$

The function $g^{(n)}(\tau)$ is a random parametric excitation. This function is assumed to be a zero mean, stationary Gaussian process with smooth spectral density $S_f^{(n)}$ up to the same frequency $\omega^{(n)}$ which is larger than any characteristic frequency of the system response. The mean and autocorrelation functions of $g^{(n)}(\tau)$ are

$$E\left[\frac{\sigma_0}{\rho c^2}(n\pi)^2 f(\tau)\right] = 0, \quad E\left[\frac{\sigma_0}{\rho c^2}(n\pi)^2 f(\tau)\frac{\sigma_0}{\rho c^2}(n\pi)^2 f(\tau+u)\right] = 2S_f^{(n)}\delta(u),$$

where u is dimensionless time delay and $\delta(\ )$ denotes the Dirac delta function.

3. Mean-square stability

Eq. (5) is a linear differential equation with a stochastic coefficient. In order to derive dynamic moment equations, Eq. (5) is first replaced by the following two Itô's stochastic differential equations:

$$\dot{X}_1 = X_2,$$
$$\dot{X}_2 = -2\beta X_2 - [\omega_0^{(n)}]^2 X_1 - g^{(n)}(\tau)X_1. \quad (7)$$

where $X_1 = \Phi_n,\ X_2 = \dot{\Phi}_n$. Application of Itô's differential rule yields the following system of differential equations for statistical moments:

$$\frac{\mathrm{d}E[X_1]}{\mathrm{d}\tau} = E[X_2],$$
$$\frac{\mathrm{d}E[X_2]}{\mathrm{d}\tau} = -[\omega^{(n)}]^2[X_1] - 2\beta E[X_2], \tag{8}$$

$$\frac{\mathrm{d}E[X_1^2]}{\mathrm{d}\tau} = 2E[X_1X_2],$$
$$\frac{\mathrm{d}E[X_1, X_2]}{\mathrm{d}\tau} = -[\omega^{(n)}]^2[X_1^2] - 2\beta E[X_1X_2] + E[X_2^2],$$
$$\frac{\mathrm{d}E[X_2^2]}{\mathrm{d}\tau} = 2S_f^{(n)}[X_1^2] - 2[\omega^{(n)}]^2E[X_1X_2] - 4\beta E[X_2^2]. \tag{9}$$

The necessary and sufficient conditions for mean-square stability are that all the eigenvalues of the coefficient matrix system (9) have negative real parts. These stability conditions are well known, and may be readily obtained by applying the Hurwitz criteria as

$$\beta > 0, \quad \beta > \frac{S_f^{(n)}}{2[\omega^{(n)}]^2}. \tag{10}$$

4. Lyapunov exponent and stochastic stability

The sign of the largest Lyapunov's exponent has an important role in determining the stability of the elastic system when it is subjected to parametric stochastic excitation. From the condition when the largest Lyapunov's exponent is negative, the system is almost stable; if it is positive, the system is almost surely unstable. In the paper by Ariaratnam and Xie [10] the largest Lyapunov's exponent is determined for the elastic system whose motion is described by differential equation (5); it states that

$$\lambda_\Phi = -\beta + \frac{\int_{-\infty}^{\infty} P(u)\mathrm{e}^{-H(u)}\,\mathrm{d}u \int_{-\infty}^{u} \mathrm{e}^{H(v)}\,\mathrm{d}v}{\int_{-\infty}^{\infty} \mathrm{e}^{-H(u)}\,\mathrm{d}u \int_{-\infty}^{u} \mathrm{e}^{H(v)}\,\mathrm{d}v}, \tag{11}$$

where

$$P(u) = \frac{(1-\gamma_0)u}{(1+u^2)} + \frac{(1-u^2)S_f^{(n)}}{(1+u^2)^2}, \quad H(u) = \frac{1}{3S_f^{(n)}}(3\gamma_0 u + u^3), \quad \gamma_0 = -\beta^2 + [\omega^{(n)}]^2.$$

The integrals in relation (11) cannot be calculated explicitly. By applying asymptotic development to the integrals in Eq. (11), Ariaratnam and Xie determined the asymptotic value for Lyapunov's exponents λ_{Φ_n} in the first order approximation, taking only the first terms in the asymptotic evaluation, when $S_f^{(n)} \to 0$

$$\lambda_{\Phi_n} = -\beta + \frac{S_f^{(n)}}{\left(1+\sqrt{\gamma_0}\right)^2}, \quad \gamma_0 > 0,$$

$$\lambda_{\Phi_n} = -\beta + \sqrt{-\gamma_0} + \frac{S_f^{(n)}(1+\gamma_0)}{(1-\gamma_0)^2}, \quad \gamma_0 < 0. \tag{12}$$

By applying asymptotic method Perevala [11], values for Lyapunov's exponents λ_{Φ_n} were obtained in the first order approximation, which is the same as the one obtained by the other approximate method (12), by Ariaratnam and Xie [10]. The same asymptotic method in the second order approximation, taking the second term in asymptotic evaluation, the next value for Lyapunov's exponents λ_{Φ_n} was obtained, when $S_f^{(n)} \to 0$

$$\lambda_{\Phi_n} = -\beta + \frac{S_f^{(n)}}{\left(1+\sqrt{\gamma_0}\right)^2}\left[1 + \frac{\left(1+\sqrt{\gamma_0}\right)(1+3\gamma_0) - 8\gamma_0}{4\gamma_0\left(1-\sqrt{\gamma_0}\right)}\right], \quad \gamma_0 > 0,$$

$$\begin{aligned}\lambda_{\Phi_n} = &-\beta + \sqrt{-\gamma_0} + \frac{S_f^{(n)}(1+\gamma_0)}{(1-\gamma_0)^2} + \frac{4\sqrt{S_f^{(n)}}(-\gamma_0)^{1/4}}{(1-16\gamma_0)} \\ &\times \left[\frac{1+\gamma_0}{1-\gamma_0} - 2S_f^{(n)}\frac{(3+\gamma_0)\sqrt{-\gamma_0}}{(1-\gamma_0)^3}\right], \quad \gamma_0 < 0.\end{aligned} \tag{13}$$

If one uses relations (13) for Lyapunov's exponent, the conditions for almost sure stochastic stability of torsion oscillations of the moving thin elastic band can be determined in the second order approximation

$$\beta > \frac{S_f^{(n)}}{\left(1+\sqrt{\gamma_0}\right)^2}\left[1 + \frac{\left(1+\sqrt{\gamma_0}\right)(1+3\gamma_0) - 8\gamma_0}{4\gamma_0\left(1-\sqrt{\gamma_0}\right)}\right], \quad \gamma_0 > 0,$$

$$\begin{aligned}\beta > &\sqrt{-\gamma_0} + \frac{S_f^{(n)}(1+\gamma_0)}{(1-\gamma_0)^2} + \frac{4\sqrt{S_f^{(n)}}(-\gamma_0)^{1/4}}{(1-16\gamma_0)} \\ &\times \left[\frac{1+\gamma_0}{1-\gamma_0} - 2S_f^{(n)}\frac{(3+\gamma_0)\sqrt{-\gamma_0}}{(1-\gamma_0)^3}\right], \quad \gamma_0 < 0.\end{aligned} \tag{14}$$

5. Numerical example

As a numerical example, determine the conditions for stochastic stability of torsion oscillations of a moving thin elastic band whose characteristics are stated by Ariaratnam and Asokanthan [4].

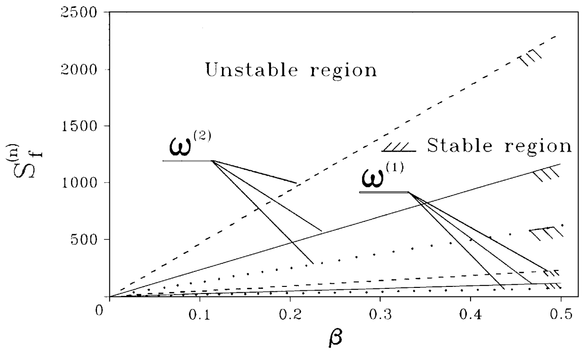

Fig. 2. Stochastic stability regions of torsional oscillations of a moving thin elastic band. ——, Mean-square stability; ····, stability region determined by using Lyapunov exponents in a first approximation; - - - -, stability region determined by using Lyapunov exponents in a second approximation.

Band thickness $b = 1.4$ mm.
Band width $h = 228.6$ mm.
Band length between supports $L = 914$ mm.
Young's modulus $E = 2.05 \times 10^{11}$ N m^{-2}.
The Poisson ratio $\upsilon = 0.3$
Mass density $\rho = 7800$ kg m^{-3}.
Speed of moving band $c = 35$ m s^{-1}.
Initial band tension $\sigma_0 = 5 \times 10^7$ N m^{-2}.

The values of the first two non-dimensional eigenfrequencies of torsion oscillations $\omega^{(1)}$ and $\omega^{(2)}$ are calculated by using relation (6); for the above-given band parameters values 10.63 and 34.06 follow in order. Fig. 2 shows particular sections of stochastic stability of torsion oscillations of the moving thin elastic band by applying the moment equations method and by using Lyapunov's coefficient, determined by the asymptotic method.

6. Conclusion

The regions of stochastic stability of torsional oscillations of a moving thin elastic band are determined when it is subjected to a random axially tensile force, which is a stochastic process of small intensity. These regions are determined by using the moment equations method and by the largest Lyapunov's exponent. The value of Lyapunov's exponent is determined by taking the second term in asymptotic evaluation to the integrals (11). The obtained results show that the region of stochastic stability is largest when it is determined on the basis of Lyapunov's exponent in a second order approximation. Notice that a larger stability region is obtained by the moment equations method comparable to the one determined by using Lyapunov's exponent in the first approximation. Likewise, notice that, at higher eigenfrequencies of the system, it has a larger stability zone determined by both the methods.

Acknowledgements

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